

Three Dimensional Origin of AdS₂ Quantum Gravity

Alejandra Castro^{◇1}, *Cynthia Keeler*^{‡2} and *Finn Larsen*^{†3}

[◇]Physics Department, McGill University, Montreal, QC H3A 2T8, Canada.

[‡]Department of Physics, Harvard University, Cambridge, MA 02138, USA.

[†]Michigan Center for Theoretical Physics, Ann Arbor, MI 48109, USA.

Abstract

We study AdS₂ quantum gravity with emphasis on consistency with results from AdS₃. We lift AdS₂ black holes to three dimensions and map fluctuations around the solutions. Comparison with near extremal BTZ are discussed, with due emphasis on global aspects. The results confirm that parameters like central charges and conformal weights computed directly in 2D are consistent with standard results in 3D. Applying our results to the thermodynamics of near extreme Kerr black holes, we show that AdS₂ quantum gravity gives the correct central charge $c = 12J$, and the entropy of excitations above the extremal limit is captured correctly.

¹ acastro@physics.mcgill.ca

² cakeeler@physics.harvard.edu

³ larsenf@umich.edu

1. Introduction

An illuminating way to analyze black hole dynamics is to focus on the 2D effective theory that governs the near horizon region after dimensional reduction of the angular directions. In extremal and near extremal settings this reasoning leads one to consider AdS_2 quantum gravity as a theory that encodes universal features of black holes near their ground state [1,2]. The expectation from holography is that this theory is dual to a 1D CFT. Indeed, AdS_2 quantum gravity is independently interesting as the most bare bones example of AdS/CFT correspondence.

However, many aspects of AdS_2 quantum gravity remain puzzling. Indeed, propagating degrees of freedom are absent in 2D gravity quite generally. A related fact is the existence of diffeomorphisms between AdS_2 black holes and global AdS_2 . These features indicate a particularly strong form of holography, where all degrees of freedom are localized on the boundary. Accordingly, the dynamics of the theory is necessarily relatively subtle, as it depends on the details of boundary conditions satisfied by diffeomorphisms.

Our approach to AdS_2 quantum gravity implements relatively conventional AdS/CFT methods (see *e.g.* [3,4,5,6] and references within), following [7,8]. The emphasis in this paper is to ensure that the subtleties encountered in 2D have been addressed correctly by embedding the theory into AdS_3 and establishing consistency with well-known 3D results. Locally, our embedding is valid exactly, with no limit needed. However, in order to satisfy the correct global identifications, and recover the $SL(2, R)$ isometry of AdS_2 from BTZ, we take a further limit. We maintain excitations above the extremal ground state in this limit, in contrast to discussions invoking a *strict* extremal limit of the BTZ black hole (see *eg.* [1,2,9,10]). This is significant because we interpret AdS_2 quantum gravity as the dynamical theory of the *excitations* above the extremal limit. In contrast, the ground state degeneracy of the extremal state is inert, because it corresponds to the direction removed by the reduction from 3D to 2D. Other interesting approaches to AdS_2 quantum gravity include [11,12,13,14,15,16,17].

Our reduction from AdS_3 to AdS_2 also extends off-shell. The restriction to fluctuations respecting asymptotically AdS_2 boundary conditions imposes a constraint that removes one chiral sector from the 3D theory. The other sector can be identified with AdS_2 quantum gravity. We verify that the central charges from these two points of view match. We also find that the conformal weight of the BTZ black hole near extremality matches that of the AdS_2 black hole.

A significant motivation for our work is the application of AdS_2 quantum gravity to the Kerr black hole. Near extreme Kerr black holes exhibit an AdS_2 near horizon region, although not one that descends from AdS_3 . Fortunately our implementation of AdS_2 quantum gravity is meaningful intrinsically in 2D — consistency with AdS_3 is ultimately for guidance and validation only. Applying our results to near extreme Kerr black hole we recover the central charge $c = 12J$ that was previously computed using other methods [18] (see also [19,20,21]). We also determine the relation between excitation energy and conformal weight. Taken together, these results give an entropy of the dual CFT which is found to agree with the area law of near extreme Kerr black holes.

It is an important feature of AdS_3 that symmetries *guarantee* agreement between the Bekenstein-Hawking entropy of BTZ black holes and the entropy of the dual CFT_2 . The key observation is that BTZ black holes are related to thermal AdS_3 by a large coordinate transformation which reduces to a modular transformation on the boundary. The spirit of our investigation is that this result should reduce to AdS_2 quantum gravity and we include preliminary evidence in this direction in the final discussion.

This paper is organized as follows. In section 2 we develop AdS_2 quantum gravity directly in 2D using standard AdS/CFT correspondence, primarily following [7,8]. We review the argument that AdS_2 black holes constitute nontrivial states at the quantum level [22,23]. In section 3 we embed our 2D theory into AdS_3 quantum gravity and show consistency with well established local results from that setting. In section 4, we match the global properties of AdS_2 black holes to those of BTZ in a suitable near horizon limit. In section 5 we apply our results to near extreme Kerr black holes.

2. AdS_2 Quantum Gravity

In this section we first review some results of AdS_2 quantum gravity and cast them in a robust form that can be readily compared to 3D. In addition, we show that the non-extremal solutions are related to the vacuum via a density matrix at the boundary.

2.1. Theory and solutions

The 2D effective theory contains a metric $g_{\mu\nu}$ with $\mu, \nu = 1, 2$, a gauge connection \mathcal{B}_μ encoding rotation or electric charges of the higher dimensional black hole and one scalar field ψ that couples to the size of the angular directions. The precise coupling among the fields in the action will depend on the details of the setting. For definiteness we focus on

the effective theory for the near extremal Kerr black hole derived in [8]. In this case the bulk action is

$$S_{\text{Kerr}} = \frac{\ell^2}{4G_4} \int d^2x \sqrt{-g} \left[e^{-2\psi} R^{(2)} + \frac{1}{\ell^2} + 2\nabla_\mu e^{-\psi} \nabla^\mu e^{-\psi} - \frac{\ell^2}{2} e^{-4\psi} \mathcal{G}^2 \right] , \quad (2.1)$$

where the gauge field strength is $\mathcal{G} = d\mathcal{B}$. Effective actions for other black holes than Kerr will differ from (2.1) by field redefinitions and overall normalization of the action. The discussion in the remainder of this paper applies to a general 2D theory of gravity with AdS_2 boundary conditions supported by a Maxwell field. The field ψ will be constant due to the AdS_2 boundary conditions [7]. The main reference to the Kerr effective action (2.1) will be that expressions involve the coupling G_4 . An alternative notation that exclusively refers to 2D physics amounts to the identification

$$G_4 = 4\pi G_2 \ell^2 e^{-2\psi} , \quad (2.2)$$

where the 2D gravitational coupling G_2 in turn may be recast in terms of the coupling appropriate to the context at hand.

Solutions with constant ψ are locally AdS_2 with radius ℓ . The equations of motion reduce to

$$R^{(2)} = -e^{2\psi} \frac{2}{\ell^2} , \quad \mathcal{G}^2 = -e^{2\psi} \frac{2}{\ell^4} , \quad \nabla_\mu \mathcal{G}^{\mu\nu} = 0 . \quad (2.3)$$

In a gauge where $\mathcal{B}_\rho = 0$ the general solution takes the form

$$ds^2 = e^{-2\psi} d\rho^2 + h_{tt} dt^2 , \quad \mathcal{B} = \mathcal{B}_t dt . \quad (2.4)$$

Then the black hole solutions to (2.3) are

$$\begin{aligned} h_{tt} &= -\frac{1}{4} e^{-2\psi} e^{2\rho/\ell} \left(1 - \frac{\epsilon^2}{\ell^2} e^{-2\rho/\ell} \right)^2 , \\ \mathcal{B}_t &= \frac{1}{2\ell} e^{\rho/\ell} \left(1 - \frac{\epsilon}{\ell} e^{-\rho/\ell} \right)^2 . \end{aligned} \quad (2.5)$$

where ϵ is the scale that controls the energy above extremality. The near extremal Kerr example (reviewed in Appendix A) amounts to the identification

$$\begin{aligned} e^{-2\psi} &= 1 , \\ \ell^2 &= 2G_4 J . \end{aligned} \quad (2.6)$$

The renormalized action for AdS₂ gravity can be obtained using the standard procedure for AdS/CFT [2,7,24]. The fundamental requirement that the variational principle must be well defined determines the boundary action such that the on-shell action is finite. For (2.1) the counterterms are

$$S_{\text{bndy}} = \frac{\ell^2}{2G_4} \int dt \sqrt{-h} \left[e^{-2\psi} K^{(2)} - \frac{1}{2\ell} e^{-\psi} + \frac{\ell}{2} e^{-3\psi} \mathcal{B}_a \mathcal{B}^a \right] , \quad (2.7)$$

with $K^{(2)}$ the extrinsic curvature at the boundary

$$K^{(2)} = \frac{1}{2} h^{tt} n^\mu \partial_\mu h_{tt} , \quad n^\rho = \sqrt{g^{\rho\rho}} . \quad (2.8)$$

2.2. Boundary conditions and asymptotic symmetries

It is essential to note that only a combination of diffeomorphisms and gauge transformations are consistent with the gauge conditions imposed on the solutions [25]. Here we will review the allowed transformations that asymptotically preserve AdS₂ boundary conditions.

The unspecified fields in the general solution (2.4) can be expanded asymptotically in the Fefferman-Graham form

$$\begin{aligned} h_{tt} &= h^{(0)} e^{2\rho/\ell} + h^{(2)} + h^{(4)} e^{-2\rho/\ell} + \dots , \\ \mathcal{B}_t &= \mathcal{B}^{(0)} e^{\rho/\ell} + \mathcal{B}^{(2)} + \mathcal{B}^{(4)} e^{-\rho/\ell} + \dots . \end{aligned} \quad (2.9)$$

The scalar is constant and set to $\psi = 0$. Solutions that are asymptotically AdS₂ satisfy the constraint

$$h^{(0)} + \ell^2 \mathcal{B}^{(0)} \mathcal{B}^{(0)} = 0 . \quad (2.10)$$

This condition expresses the fact that the field strength \mathcal{G} must be proportional to the AdS₂ volume two-form. On-shell we can specify the leading behavior as

$$\mathcal{B}^{(0)} = \frac{1}{2\ell} , \quad h^{(0)} = -\frac{1}{4} . \quad (2.11)$$

The remaining terms in (2.9), $h^{(n)}$ and $\mathcal{B}^{(n)}$ with $n = 2, 4, \dots$, are arbitrary functions of time. Notice that $\mathcal{B}^{(2)}$ is pure gauge, and its value is fixed by demanding regularity of the solution.

Under diffeomorphisms $x^\mu \rightarrow x^\mu + \epsilon^\mu$ with parameter

$$\begin{aligned} \epsilon^t &= \xi(t) - \frac{\ell^2}{2h^{(0)}} \partial_t^2 \xi e^{-2\rho/\ell} + \mathcal{O}(e^{-4\rho/\ell}) , \\ \epsilon^\rho &= -\ell \partial_t \xi(t) , \end{aligned} \quad (2.12)$$

the gauge conditions for the metric, *i.e.* $g_{\rho\rho} = 1$ and $g_{\rho t} = 0$, are preserved. However, such diffeomorphisms transform the gauge field such that $\mathcal{B}_\rho \neq 0$. To restore the gauge condition $\mathcal{B}_\rho = 0$ we compensate the diffeomorphism by a gauge transformation with parameter

$$\Lambda = -\frac{\ell}{h^{(0)}} \partial_t^2 \xi \left(\mathcal{B}^{(0)} e^{\rho/\ell} + \frac{1}{2} \mathcal{B}^{(2)} \right) e^{-2\rho/\ell} + \mathcal{O}(e^{-3\rho/\ell}) . \quad (2.13)$$

Combining the diffeomorphism (2.12) and gauge transformation (2.13) preserves our asymptotic boundary conditions. The metric and gauge fields transform as

$$\begin{aligned} \delta_\epsilon h_{tt} &= \xi \partial_t h^{(2)} - 2\partial_t \xi h^{(2)} - \ell^2 \partial_t^3 \xi + \mathcal{O}(e^{-2\rho/\ell}) , \\ \delta_{\epsilon+\Lambda} \mathcal{B}_t &= \partial_t (\xi \mathcal{B}^{(2)}) + \left[\xi \partial_t \mathcal{B}^{(4)} - \partial_t \xi \mathcal{B}^{(4)} \right] e^{-\rho/\ell} + \frac{\ell^2 \mathcal{B}^{(0)}}{2h^{(0)}} \partial_t^3 \xi e^{-\rho/\ell} + \mathcal{O}(e^{-2\rho/\ell}) . \end{aligned} \quad (2.14)$$

2.3. Conserved charges

There are two relevant transformations in our discussion, diffeomorphisms and gauge transformations. For each of them there is a corresponding generator (a boundary current) which we want to construct.

Generator of diffeomorphisms. The variation of the action under a diffeomorphism is given by

$$\delta_\epsilon S = \int dt \sqrt{-h} \left[\frac{1}{2} T^{ab} \delta_\epsilon h_{ab} + \sqrt{-h} J^a \delta_\epsilon \mathcal{B}_a \right] + (\text{e.o.m}) , \quad (2.15)$$

where

$$\begin{aligned} \delta_\epsilon h_{ab} &= \nabla_a \epsilon_b + \nabla_b \epsilon_a , \\ \delta_\epsilon \mathcal{B}_a &= \epsilon^b \nabla_b \mathcal{B}_a + \mathcal{B}_b \nabla_a \epsilon^b , \end{aligned} \quad (2.16)$$

and

$$\begin{aligned} T_{tt} &= -\frac{\ell^2}{4G_4} \left(\frac{1}{\ell} e^{-\psi} h_{tt} + \ell e^{-3\psi} \mathcal{B}_t \mathcal{B}_t \right) , \\ J_t &= \frac{\ell^2}{2G_4} e^{-3\psi} \left(-e^{-\psi} \ell^2 n^\mu \mathcal{G}_{\mu t} + \ell \mathcal{B}_t \right) . \end{aligned} \quad (2.17)$$

Inserting the Fefferman-Graham expansion (2.9) these currents become

$$\begin{aligned} T_{tt} &= -\frac{\ell}{4G_4} \left[2\ell^2 \mathcal{B}^{(0)} \mathcal{B}^{(2)} + (h^{(2)} + \ell^2 \mathcal{B}^{(2)} \mathcal{B}^{(2)} + 2\ell^2 \mathcal{B}^{(0)} \mathcal{B}^{(4)}) e^{-\rho/\ell} + \mathcal{O}(e^{-2\rho/\ell}) \right] e^{\rho/\ell} , \\ J_t &= \frac{\ell^2}{2G_4} \left[\mathcal{B}^{(2)} + 2\mathcal{B}^{(4)} e^{-\rho/\ell} + \mathcal{O}(e^{-2\rho/\ell}) \right] . \end{aligned} \quad (2.18)$$

Usually the generator of diffeomorphisms is just the stress tensor T_{ab} because the matter terms in (2.15) decay more rapidly than the metric field. In 2D the situation is

different since the rapid increase $\mathcal{B}_t \sim e^{\rho/\ell}$ at the boundary is needed to support the solution. This situation imposes the constraint (2.10) between the leading terms in the Fefferman-Graham expansion (2.9), which in turn will imply that the two terms in (2.15) are of the same order. Explicitly, to preserve the asymptotic AdS₂ boundary conditions the variations of metric and gauge field must be related as

$$\delta h^{(0)} = -2\ell^2 \mathcal{B}^{(0)} \delta \mathcal{B}^{(0)} . \quad (2.19)$$

so that (2.15) becomes

$$\begin{aligned} \delta_\epsilon S &= \int dt \sqrt{-h} \left(-\ell^2 \mathcal{B}^{(0)} T^{tt} e^{\rho/\ell} + J^t \right) e^{\rho/\ell} \delta \mathcal{B}^{(0)} \\ &= \frac{2\ell^2}{G_4} \int dt \sqrt{-h} \left[h^{(2)} + \ell^2 \mathcal{B}^{(2)} \mathcal{B}^{(2)} - 2\ell^2 \mathcal{B}^{(0)} \mathcal{B}^{(4)} + \mathcal{O}(e^{-\rho/\ell}) \right] e^{-2\rho/\ell} \delta \mathcal{B}^{(0)} \\ &= \frac{2\ell^2}{G_4} \int dt \sqrt{-h} \ell^2 \mathcal{B}^{(2)} \mathcal{B}^{(2)} e^{-2\rho/\ell} \delta \mathcal{B}^{(0)} . \end{aligned} \quad (2.20)$$

In the second line we used the currents (2.18) and the on-shell values (2.11). In the last line we used the value $h^{(2)} = 2\ell^2 \mathcal{B}^{(0)} \mathcal{B}^{(4)}$ imposed by the equations of motion, whose perturbative expansion is detailed in Appendix B.

The time translation operator is usually interpreted as the energy of the configuration. Accordingly, (2.20) assigns an energy of the form $E \sim \mathcal{B}^{(2)2}$ to excitations in the theory. To avoid unnecessary dependence on conventions we will not specify the proportionality constant in the energy, but just work with variations of the action. We are especially interested in the specific case of the AdS₂ black hole (2.5) for which

$$\delta_\epsilon S = \frac{2}{G_4} \int dt \sqrt{-h} \epsilon^2 e^{-2\rho/\ell} \delta \mathcal{B}^{(0)} . \quad (2.21)$$

In the next section we will recover this generator from the 3D perspective.

Generator of gauge transformations. The variation of the action under a gauge transformation is

$$\delta_\Lambda S = \int dt \sqrt{-h} \mathcal{J}^a \delta_\Lambda \mathcal{B}_a , \quad (2.22)$$

where

$$\mathcal{J}_t = \frac{\ell^3}{2G_4} e^{-3\psi} \mathcal{B}_t . \quad (2.23)$$

Allowed gauge transformations correspond to variations of the action with respect to the constant term in the gauge field $\mathcal{B}^{(2)}$ and not $\mathcal{B}^{(0)}$, the leading term⁴. Using our asymptotic expansion (2.9), the generator (2.22) reads

$$\begin{aligned}\delta_\Lambda S &= -\frac{2\ell^3}{G_4} \int dt \sqrt{-h} \mathcal{B}^{(0)} e^{-\rho/\ell} \delta_\Lambda \mathcal{B}^{(2)} \\ &= -\frac{\ell^2}{G_4} \int dt \sqrt{-h} e^{-\rho/\ell} \delta_\Lambda \mathcal{B}^{(2)} .\end{aligned}\tag{2.24}$$

Since $\sqrt{-h} \sim e^{\rho/\ell}$, the allowed gauge transformations modify the action by at most a constant term [7].

The generator of gauge transformations (2.24) can be interpreted as the charge of the electric field that supports the solution [8]. Our interest will be in comparing with the corresponding 3D result and for that purpose the variation (2.24) is a convenient and unambiguous expression for the charge.⁵

2.4. Thermal states in AdS_2

An important issue to address is if the solutions (2.4) - (2.5) to the 2D theory correspond to distinct states of the dual theory or if they are trivially related by a diffeomorphism. To explore this question we will see how an AdS_2 black hole ($\epsilon \neq 0$) is related to the vacuum solution ($\epsilon = 0$). It is convenient to write (2.4) as

$$ds^2 = -\frac{U^2 - \epsilon^2}{\ell^2} dt^2 + \frac{\ell^2}{U^2 - \epsilon^2} dU^2 , \quad \mathcal{B} = \frac{U - \epsilon}{\ell^2} dt .\tag{2.25}$$

where the coordinate U is related to the radial coordinate in (2.5) as

$$U = \frac{\ell}{2} e^{\rho/\ell} \left(1 + \frac{\epsilon^2}{\ell^2} e^{-2\rho/\ell} \right) .\tag{2.26}$$

The value of the modified stress tensor (2.20) evaluated on (2.25) is

$$T_{tt} + \mathcal{B}_t J_t = \frac{\ell^3}{4G_4} \mathcal{B}^{(2)2} = \frac{\ell}{4G_4} \frac{\epsilon^2}{\ell^2} .\tag{2.27}$$

⁴ We specify the gauge $\mathcal{B}_\rho = 0$ so residual gauge transformations are independent of ρ , but they may depend on the boundary coordinate t .

⁵ In situations where the gauge transformation is accompanied by a diffeomorphism the transformation of $\mathcal{B}^{(2)}$ is dominated by the gauge current $\mathcal{J}_t \sim \mathcal{O}(e^{\rho/\ell})$ while $J^t \sim \mathcal{O}(1)$.

The vacuum solution is AdS_2 in Poincare coordinates

$$ds^2 = \frac{\ell^2}{y^2}(dy^2 - dw^2) , \quad \mathcal{B} = -\frac{\ell}{y}dw . \quad (2.28)$$

The charge associated to diffeomorphisms for this background is zero, *i.e.* $T_{tt} + \mathcal{B}_t J_t = 0$.

The transformation that relates the black hole (2.25) and (2.28) is given by the coordinate change

$$\begin{aligned} y &= \frac{\ell\epsilon}{\sqrt{U^2 - \epsilon^2}} e^{-\epsilon t/\ell^2} , \\ w &= \ell \frac{U}{\sqrt{U^2 - \epsilon^2}} e^{-\epsilon t/\ell^2} , \end{aligned} \quad (2.29)$$

accompanied by the gauge transformation

$$\Lambda = -\frac{1}{2} \ln \left(\frac{U + \epsilon}{U - \epsilon} \right) . \quad (2.30)$$

There are a couple of things worth mentioning here. First, the coordinate transformation (2.29) is singular at the horizon $U = \epsilon$. Second, near the boundary $U \rightarrow \infty$ we have

$$w = \ell e^{-\epsilon t/\ell^2} + \mathcal{O}(U^{-1}) . \quad (2.31)$$

This relation mimics that between Rindler and Minkowski coordinates. In that case, the quantum states are related by a density matrix such that the Rindler observer will detect a thermal bath with temperature

$$T = \frac{\epsilon}{2\pi\ell^2} . \quad (2.32)$$

The modified boundary stress tensor $T_{tt} + \mathcal{B}_t J_t$ transforms as an affine tensor in the usual manner so the transformation (2.31) induces the anomalous energy

$$\begin{aligned} T_{tt} + \mathcal{B}_t J_t &= \frac{c}{12} \ell \{w, t\} (\partial_w t)^{-2} \\ &= \frac{\pi^2}{6} c \ell T^2 , \end{aligned} \quad (2.33)$$

where the Schwarzian derivative

$$\{w, t\} = \frac{\partial_w^3 t}{\partial_w t} - \frac{3}{2} \left(\frac{\partial_w^2 t}{\partial_w t} \right)^2 = \frac{1}{2w^2} . \quad (2.34)$$

Comparing with (2.27) found by evaluation on the black hole solution we deduce the central charge

$$c = \frac{6\ell^2}{G_4} . \quad (2.35)$$

This is the same result found in [8], by examination of the transformation properties of the modified boundary stress tensor.

Note that in (2.33) we are measuring energies with respect to the AdS_2 radius ℓ . This scale is a natural one to use in the 2D theory, but arbitrary since the theory is exactly conformal. In section 5, when comparing with the Kerr black hole (which breaks conformal invariance), we will compute the scale R and verify that energies near extremality agree after an appropriate rescaling.

In summary, we have exhibited a classical diffeomorphism that relates the non-extremal AdS_2 metric ($\epsilon \neq 0$) to the extremal one. In the quantum theory the transformation at the boundary induces a thermal state due to a scale anomaly. Thus the two AdS_2 metrics correspond to different quantum states.

3. AdS_2 Quantum Gravity from 3D Gravity

In this section we provide a map between 2D solutions and the 3D configurations. We show that the boundary currents in 3D reproduce the same values as those found in 2D. Finally, we lift the asymptotic symmetry group in 2D and find the effects on the 3D stress tensor. This will allow us to identify the central charges in 3D and 2D.

3.1. Reduction of the stress tensor

We start by showing how the generators of diffeomorphisms and charges in AdS_2 quantum gravity can be recovered from the 3D stress tensor. The starting point is three dimensional gravity described by the action

$$I = \frac{1}{16\pi G_3} \int d^3x \sqrt{-g} \left(R^{(3)} + \frac{2}{L^2} \right) + \frac{1}{8\pi G_3} \int d^2y \sqrt{-\gamma} \left(K^{(3)} - \frac{1}{L^2} \right), \quad (3.1)$$

where we have included the bulk and boundary actions. The stress tensor $T_{3\text{D}}^{ab}$ is defined via

$$\delta I = \frac{1}{2} \int d^2y \sqrt{-\gamma} T_{3\text{D}}^{ab} \delta \gamma_{ab}. \quad (3.2)$$

This definition gives

$$T_{ab}^{3\text{D}} = \frac{1}{8\pi G_3} \left(K_{ab}^{(3)} - K^{(3)} \gamma_{ab} + \frac{1}{L} \gamma_{ab} \right), \quad (3.3)$$

where $K_{ab}^{(3)}$ is the extrinsic curvature. Here y^a with $a = 1, 2$ are the boundary coordinates, and the metric γ_{ab} is defined as

$$ds_3^2 = L^2 d\eta^2 + \gamma_{ab} dy^a dy^b . \quad (3.4)$$

In the 3D gravity, the boundary metric has a Fefferman-Graham expansion of the form

$$\gamma_{ab} = e^{2\eta} \gamma_{ab}^{(0)} + \gamma_{ab}^{(2)} + e^{-2\eta} \gamma_{ab}^{(4)} + \dots , \quad (3.5)$$

where the dots correspond to higher powers of $e^{-2\eta}$.

We need to cast (3.2) as a variation with respect to the 2D fields h_{tt} and \mathcal{B}_t instead of γ_{ab} . This will allow us to express the 2D currents discussed in the previous section in terms of T_{ab}^{3D} . We will use the Kaluza-Klein reduction from 3D to 2D, given by

$$ds^2 = \ell^2 (d\theta + \mathcal{B}_t dt)^2 + d\rho^2 + h_{tt} dt^2 . \quad (3.6)$$

Inserting the 2D black holes (2.5), we find that the 3D lift (3.6) becomes

$$ds_3^2 = d\rho^2 + \ell e^{\rho/\ell} \left(1 - \frac{\epsilon}{\ell} e^{-\rho/\ell}\right)^2 \left[dt d\theta - \frac{\epsilon}{\ell^2} dt^2\right] + \ell^2 d\theta^2 . \quad (3.7)$$

We would like to rewrite this metric in coordinates natural from the 3D point of view; that is, we want to rewrite (3.7) in the form (3.4). Comparing (3.7) with (3.4) and (3.5) we get

$$\rho = L\eta , \quad 2\ell = L . \quad (3.8)$$

We can continue to use (t, θ) to describe the remaining two dimensions, but eventually we would like to compare to light-like coordinates natural for 3D boundary. If we describe the boundary of AdS_3 as

$$\gamma_{ab}^{(0)} dx^a dx^b = L^2 dx^+ dx^- , \quad (3.9)$$

then from the leading term in (3.7) we find

$$\begin{aligned} x^- &= \frac{t}{2\ell} , \\ x^+ &= \frac{\theta}{2} - \frac{\epsilon}{2\ell^2} t . \end{aligned} \quad (3.10)$$

For now, we will continue in the combined coordinates (η, t, θ) . Additionally, for sake of simplicity, throughout this section we set the dilaton to the on-shell value $\psi = 0$. We

now write γ_{ab} in terms of h_{tt} and \mathcal{B}_t . From (3.6) the boundary metric in (t, θ) coordinates is

$$\gamma_{tt} = h_{tt} + \ell^2 \mathcal{B}_t \mathcal{B}_t, \quad \gamma_{t\theta} = \ell^2 \mathcal{B}_t, \quad \gamma_{\theta\theta} = \ell^2. \quad (3.11)$$

Using the 2D Fefferman-Graham expansion (2.9) in (3.11) we get

$$\begin{aligned} \gamma_{tt} &= \left[h^{(0)} + \ell^2 \mathcal{B}^{(0)} \mathcal{B}^{(0)} \right] e^{4\eta} + 2\ell^2 \mathcal{B}^{(2)} \mathcal{B}^{(0)} e^{2\eta} \\ &\quad + \left[h^{(2)} + \ell^2 \mathcal{B}^{(2)} \mathcal{B}^{(2)} + 2\ell^2 \mathcal{B}^{(0)} \mathcal{B}^{(4)} \right] + \dots, \\ \gamma_{t\theta} &= \ell^2 \mathcal{B}^{(0)} e^{2\eta} + \ell^2 \mathcal{B}^{(2)} + \ell^2 \mathcal{B}^{(4)} e^{-2\eta} + \dots, \\ \gamma_{\theta\theta} &= \ell^2, \end{aligned} \quad (3.12)$$

where we used $\rho/\ell = 2\eta$ according to (3.8).

Before performing the reduction, we should be more precise about how to manipulate and interpret the variation of the action (3.2) and the resultant stress tensor (3.3). In (3.5) $\gamma^{(0)}$ is the boundary metric up to a conformal transformation and the sub-leading terms contain information about the mass and angular momentum of the solution. According to the AdS/CFT dictionary, the stress tensor is obtained via the variation of the action with respect to the boundary metric

$$\delta\gamma_{ab} = e^{2\eta} \delta\gamma_{ab}^{(0)}. \quad (3.13)$$

Thus, it is the variation of the action with respect to $\gamma^{(0)}$ which we will be interested in comparing to the 2D results. Of course, the stress tensor itself will still be as given in (3.3); we need only make the replacement (3.13) when writing the expression for the variation of the action.

There is an additional subtlety which we must address. At first glance, the metric in (3.12) indicates leading behavior $\gamma_{tt} \sim e^{4\eta}$ which is not the correct asymptotic behavior. However, we restrict to asymptotically AdS₂ geometry by imposing the constraint (2.10) which forces the first term in (3.12) for γ_{tt} to vanish. Thus the variation of the 3D boundary metric (3.13) by changes relevant on the 2D boundary is

$$\begin{aligned} \delta\gamma_{tt} &= 2\ell^2 \mathcal{B}^{(2)} e^{2\eta} \delta\mathcal{B}^{(0)} + 2\ell^2 \mathcal{B}^{(0)} e^{2\eta} \delta\mathcal{B}^{(2)}, \\ \delta\gamma_{t\theta} &= \ell^2 e^{2\eta} \delta\mathcal{B}^{(0)}, \\ \delta\gamma_{\theta\theta} &= 0. \end{aligned} \quad (3.14)$$

We re-write the variation of the 3D action (3.2) using the $\gamma^{(0)}$ component of (3.14) which gives

$$\delta I = \int d^2 y \sqrt{-h} \ell^2 e^{2\eta} \left[\mathcal{B}^{(2)} T_{3D}^{tt} + T_{3D}^{t\theta} \right] \delta \mathcal{B}^{(0)} + \int d^2 y \sqrt{-h} \ell^2 e^{2\eta} \left[\mathcal{B}^{(0)} T_{3D}^{tt} \right] \delta \mathcal{B}^{(2)} . \quad (3.15)$$

The next step is to compute the 3D stress tensor given by (3.3) as a function of 2D fields $h^{(n)}$ and $\mathcal{B}^{(n)}$. The details of the calculation are presented in appendix B, and the final result for the components of T_{ab}^{3D} with the indices raised is

$$\begin{aligned} T_{3D}^{tt} &= \frac{1}{8\pi G_3 L} \frac{e^{-4\eta}}{\ell^2 \mathcal{B}^{(0)} \mathcal{B}^{(0)}} \left[1 + 2e^{-4\eta} \frac{\mathcal{B}^{(4)}}{\mathcal{B}^{(0)}} + \mathcal{O}(e^{-6\eta}) \right] , \\ T_{3D}^{t\theta} &= -\frac{1}{8\pi G_3 L} \frac{e^{-4\eta}}{\ell^2 \mathcal{B}^{(0)} \mathcal{B}^{(0)}} \left[\mathcal{B}^{(2)} + 2e^{-2\eta} \mathcal{B}^{(4)} + \mathcal{O}(e^{-4\eta}) \right] . \end{aligned} \quad (3.16)$$

Separating the contributions due to $\delta \mathcal{B}^{(0)}$ and $\delta \mathcal{B}^{(2)}$ in (3.15), we find

$$\begin{aligned} \delta I_{(0)} &\equiv \int d^2 y \sqrt{-h} \ell^2 e^{2\eta} \left[\mathcal{B}^{(2)} T_{3D}^{tt} + T_{3D}^{t\theta} \right] \delta \mathcal{B}^{(0)} \\ &= -\frac{1}{G_3 \ell} \int dt \sqrt{-h} \epsilon^2 e^{-4\eta} \delta \mathcal{B}^{(0)} , \end{aligned} \quad (3.17)$$

and

$$\begin{aligned} \delta I_{(2)} &\equiv \int d^2 y \sqrt{-h} \ell^2 e^{2\eta} \left[\mathcal{B}^{(0)} T_{3D}^{tt} \right] \delta \mathcal{B}^{(2)} \\ &= \frac{\ell}{2G_3} \int dt \sqrt{-h} e^{-2\eta} \delta \mathcal{B}^{(2)} , \end{aligned} \quad (3.18)$$

where we have used (B.5) and

$$\int d^2 y = 2\pi L \int dt . \quad (3.19)$$

The $\delta I_{(0)}$ and $\delta I_{(2)}$ in these expressions should be compared with the variations of the 2D action computed in section 2.2. We find that (3.17) agrees precisely with the generator of diffeomorphisms (2.21), and (3.18) with the generator of gauge transformations (2.24):

$$\delta I_{(0)} = -\delta S_\epsilon , \quad \delta I_{(2)} = -\delta S_\Lambda , \quad (3.20)$$

after the identification

$$G_4 = 2\ell G_3 , \quad (3.21)$$

and taking into account the identification in (3.8).

As we have mentioned previously, our choice to normalize the 2D action using the 4D coupling G_4 is to facilitate comparison with Kerr. However, the identification of couplings in (3.21) clearly applies in other contexts as well, after introducing the 2D coupling through (2.2).

The map constructed in the beginning of this section determines the identification between solutions of 3D and 2D theories. The agreement (3.20) goes further, by showing an exact match of the boundary currents in the two theories.

3.2. Lift of 2D asymptotic symmetry group to 3D

In section 2.1 we constructed the asymptotic symmetry group in 2D, which includes diffeomorphisms (2.12) and gauge transformations (2.13). Here we lift these transformations with the purpose of understanding the effects of 2D asymptotic symmetries from a 3D point of view.

Start by considering the three dimensional line element

$$ds_3^2 = \ell^2 (d\theta + \mathcal{B})^2 + d\rho^2 + h_{tt} dt^2 . \quad (3.22)$$

A 2D diffeomorphism acting on (ρ, t)

$$\begin{aligned} t &\rightarrow t + \epsilon^t , \\ \rho &\rightarrow \rho + \epsilon^\rho , \end{aligned} \quad (3.23)$$

also corresponds to a diffeomorphism in 3D. On the other hand, a gauge transformation $\mathcal{B} \rightarrow \mathcal{B} + d\Lambda$ lifts to a diffeomorphism on the KK direction

$$\theta \rightarrow \theta + \Lambda . \quad (3.24)$$

Using in (3.23) and (3.24) the 2D parameters (2.12) and (2.13), the components of (3.22) transform as

$$\begin{aligned} d\theta + \mathcal{B}_t dt &\rightarrow d\theta + \mathcal{B}_t dt + \frac{\ell^2 \mathcal{B}^{(0)}}{2h^{(0)}} \partial_t^3 \xi e^{-\rho/\ell} dt + \mathcal{O}(e^{-2\rho/\ell}) , \\ h_{tt} dt^2 &\rightarrow h_{tt} dt^2 - \ell^2 \partial_t^3 \xi dt^2 + 2\ell \partial_t^2 \xi d\rho dt + \mathcal{O}(e^{-2\rho/\ell}) , \\ d\rho^2 &\rightarrow d\rho^2 - 2\ell \partial_t^2 \xi d\rho dt , \end{aligned} \quad (3.25)$$

hence the line element transforms as

$$ds_3^2 \rightarrow ds_3^2 + \left(\frac{\ell^2 \mathcal{B}^{(0)} \mathcal{B}^{(0)}}{h^{(0)}} - 1 \right) \ell^2 \partial_t^3 \xi dt^2 = ds_3^2 - 2\ell^2 \partial_t^3 \xi dt^2 , \quad (3.26)$$

where we applied the constraint (2.11) that imposes asymptotic AdS_2 boundary conditions. Thus we see that 2D diffeomorphisms generate an anomalous transformation on the 3D metric. Indeed, the additional term is exactly the anomaly for the T_{tt}^{3D} component of stress tensor in 3D. Now working in the x^\pm coordinates defined in (3.10), we have

$$t = Lx^- , \quad (3.27)$$

and so the anomalous transformation would be

$$ds_3^2 \rightarrow ds_3^2 - 2\ell^2 \partial_-^3 \xi^- (dx^-)^2 , \quad (3.28)$$

where $\xi^- = L\xi(t)$. Using (3.3) to compute the stress tensor for (3.28) we have

$$\begin{aligned} T_{--} &= -\frac{1}{8\pi G_3 L} (2\ell^2) \partial_-^3 \xi^- \\ &= -\frac{\ell}{8\pi G_3} \partial_-^3 \xi^- \\ &= -\frac{c}{24\pi} \partial_-^3 \xi^- . \end{aligned} \quad (3.29)$$

In the last line we introduced the normalization of [3] for the central charge. It implies

$$c = \frac{3\ell}{G_3} = \frac{3L}{2G_3} , \quad (3.30)$$

the Brown-Henneaux central charge. It is an important consistency check that this agrees with our 2D result for the central charge (2.35), due to the identification (3.21).

Recall that the isometry group of AdS_2 is $SL(2, R)_R \times U(1)_L$ from the 3D point of view. Because the asymptotic symmetries in 2D only affect the x^- component of the stress tensor, our result (3.30) is the right moving central charge $c = c_R$ due to an enhancement of $SL(2, R)_R$ symmetries to a Virasoro algebra. The central charge computed for extremal black holes in *e.g.* [18,10] considers an enhancement of $U(1)_L$ to a Virasoro algebra with central charge c_L . That central charge must be the same as ours, since the 3D theory is not chiral, but its origin and interpretation is different.

4. Global Aspects of the Embedding into 3D

So far we have focused on local properties of quantum gravity. We have lifted solutions of AdS_2 gravity to those of AdS_3 . However we would like to do more; we want to identify the AdS_2 black hole (which is locally AdS_2) with the BTZ black hole (which is locally AdS_3). In order to do so, we will have to discuss the *global* aspects of the embedding; we will do so in this section

4.1. The AdS_2 black hole and the BTZ solution

The BTZ black hole in Schwarzschild coordinates is

$$ds^2 = \frac{L^2 r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 - \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} dT^2 + r^2 \left(d\phi + \frac{r_+ r_-}{r^2} dT \right)^2, \quad (4.1)$$

where L is the AdS_3 radius and r_{\pm} are the outer and inner horizon coordinates. For nonextremal black holes $r_+ \neq r_-$ we can make the Fefferman-Graham (FG) expansion manifest by using the coordinates

$$r^2 = r_+^2 \cosh^2(\eta - \eta_0) - r_-^2 \sinh^2(\eta - \eta_0), \quad (4.2)$$

$$w^{\pm} = \phi \pm T, \quad (4.3)$$

with

$$e^{2\eta_0} = \frac{r_+^2 - r_-^2}{4L^2}. \quad (4.4)$$

Inserting in (4.1) we find the BTZ black hole in the FG form

$$ds^2 = L^2 d\eta^2 + \left(L^2 e^{2\eta} + \frac{1}{16L^2} (r_+^2 - r_-^2)^2 e^{-2\eta} \right) dw^+ dw^- + \frac{1}{4} (r_+ - r_-)^2 (dw^-)^2 + \frac{1}{4} (r_+ + r_-)^2 (dw^+)^2. \quad (4.5)$$

We can compare this expression with the lifted 2D black hole solution (3.7), which is similarly written in FG form. For this we rescale the radial coordinate as in (3.8) and transform the boundary coordinates linearly as

$$w^- = \frac{2\epsilon}{\ell(r_+ - r_-)} t = \frac{r_+ + r_-}{8\ell^2} t, \quad (4.6)$$

$$w^+ = \frac{2}{r_+ + r_-} \left(\ell\theta - \frac{\epsilon}{\ell} t \right) = \frac{2\ell}{r_+ + r_-} \theta - \frac{r_+ - r_-}{8\ell^2} t.$$

The parameters of the solutions are identified as

$$L = 2\ell, \quad \frac{\epsilon}{\ell} = \frac{r_+^2 - r_-^2}{16\ell^2}. \quad (4.7)$$

The 2D temporal coordinate t is identified with w^- for all the BTZ black holes, whether extremal or not. The complementary coordinate θ , parametrizing the KK circle, approaches the null coordinate w^+ in the extremal limit $r_+ \rightarrow r_-$. The parameter ϵ

describes the departure from the extremal limit, ie. the degree of non-nullness in the dimensional reduction.

The map (4.6)-(4.7) identifies the AdS₂ black hole with BTZ *locally*, but it also exhibits a *global* discrepancy. Indeed, the BTZ is defined with the identification $\phi \sim \phi + 2\pi$, corresponding to simultaneous shift of w^+ and w^- by 2π . On the other hand, the KK lifted AdS₂ black hole has

$$\theta \sim \theta + 4\pi , \quad (4.8)$$

because the length of the KK circle is $2\pi L = 4\pi\ell$. Comparing with (4.6) we see that the periodicities of the two solutions differ, as claimed.

Despite the general discrepancy, there is a near horizon limit where the identifications do coincide, as discussed in detail in [10]. The DLCQ limit is implemented by changing the coordinates as

$$w^- \rightarrow \frac{w^-}{\lambda} , \quad e^{2\eta} \rightarrow \lambda e^{2\eta} , \quad (4.9)$$

and writing the parameters of the solution as

$$r_+ - r_- = 4\lambda\epsilon . \quad (4.10)$$

After those replacements, simply take $\lambda \rightarrow 0$ with all other quantities fixed. In this limit w^- has been unwrapped so that it is no longer periodic, consistent with its identification as 2D time in the first equation in (4.6). The 2π periodicity of w^+ remains after the limit so the second equation in (4.6) imposes

$$r_+ + r_- = 4\ell , \quad (4.11)$$

for consistency with (4.8).

After the DLCQ limiting procedure the solution (4.5) reads

$$ds^2 = L^2 d\eta^2 + (L^2 e^{2\eta} + 4\epsilon^2 e^{-2\eta}) dw^+ dw^- + 4\epsilon^2 (dw^-)^2 + L^2 (dw^+)^2 , \quad (4.12)$$

and the coordinate identifications (4.6) simplify to (3.10), now in the form

$$\begin{aligned} w^- &= \frac{1}{2\ell} t \\ w^+ &= \frac{1}{2} \theta - \frac{\epsilon}{2\ell^2} t . \end{aligned} \quad (4.13)$$

The metric (4.12) is superficially similar to (4.5) but it differs in two crucial aspects. First, as we have emphasized, the scaling (4.9) unwraps the compact coordinate w^- in (4.5) so

that it becomes non-compact in (4.12). Thus (4.12) with (4.13) inserted is completely equivalent to the lifted AdS₂ black hole (3.7), including global aspects. Next, it is evident from (4.12) that a finite energy-momentum tensor should be assigned to excitations with finite ϵ *after* taking the limit. In other words, whereas the original solution (4.5), as well as the parameter ϵ in (4.7), might give the impression that the energy of excitations vanish in the extremal limit (4.10), the energy is in fact finite due to the simultaneous coordinate rescalings (4.9).

In summary, after the DLCQ limit required by the global aspects, the mass of the original 3D black hole is fixed by (4.11). The limiting theory assigns finite energy to the excitations above the extremal limit parametrized by ϵ .

4.2. AdS₂ black hole as a quotient

It is illuminating to implement the DLCQ limit in the construction of the BTZ black hole as a quotient of AdS₃.

In the region outside the horizon, the coordinate change

$$\begin{aligned} z^+ &= \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}} e^{2\pi T_+(\phi+T)} , \\ z^- &= \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}} e^{2\pi T_-(\phi-T)} , \\ y &= \sqrt{\frac{r_+^2 - r_-^2}{r^2 - r_-^2}} e^{\pi T_+(\phi+T) + \pi T_-(\phi-T)} , \end{aligned} \tag{4.14}$$

with the temperatures

$$\begin{aligned} T_+ &= \frac{r_+ + r_-}{2\pi L} , \\ T_- &= \frac{r_+ - r_-}{2\pi L} , \end{aligned} \tag{4.15}$$

maps the BTZ black hole (4.1) to Poincaré AdS₃

$$ds_3^2 = \frac{L^2}{y^2} (dy^2 + dz^+ dz^-) . \tag{4.16}$$

The 2π identification on ϕ therefore amounts to simultaneous identifications of the Poincaré coordinates

$$\begin{aligned} z^+ &\sim z^+ e^{4\pi^2 T_+} , \\ z^- &\sim z^- e^{4\pi^2 T_-} , \\ y &\sim y e^{2\pi^2 (T_- + T_+)} . \end{aligned} \tag{4.17}$$

These exponential identifications correspond to the hyperbolic/hyperbolic conjugacy class in the $SL(2, R)_L \times SL(2, R)_R$ symmetry of AdS_3 .

The analogous classification for the AdS_2 black hole follows by simplifying the temperatures (4.15) using (4.10), (4.11) and then employing the coordinates (4.13). The embedding (4.14) into Poincaré AdS_3 becomes

$$\begin{aligned} z^+ &= \tanh(\eta - \eta_0) e^{\theta - \epsilon t / \ell^2} , \\ z^- &= \tanh(\eta - \eta_0) e^{\epsilon t / \ell^2} , \\ y &= \frac{1}{\cosh(\eta - \eta_0)} e^{\theta/2} . \end{aligned} \tag{4.18}$$

The 4π identification on θ (4.8) therefore amounts to simultaneous identifications of the Poincaré coordinates

$$\begin{aligned} z^+ &\sim z^+ e^{4\pi} , \\ y &\sim y e^{2\pi} . \end{aligned} \tag{4.19}$$

These identifications correspond to hyperbolic/identity conjugacy class in the $SL(2, R)_L \times SL(2, R)_R$ symmetry of AdS_3 .

That the z^- coordinate does not participate in the identifications is due to the non-compactness of the R sector. In terms of symmetries, the hyperbolic identification in the z^+ coordinate breaks $SL(2, R)_L \rightarrow U(1)_L$, as it does for BTZ. However, the $SL(2, R)_R$ symmetry remains unbroken for the AdS_2 black hole, since there is no identification of z^- . The preserved symmetry is the isometry of the AdS_2 geometry [10].

It is instructive to compare the DLCQ limit with the usual extremal limit, where one takes $T_- \rightarrow 0$ without rescaling w^- as in (4.9). In this case w^- remains periodic, and so there is no restoration of $SL(2, R)_R$. The map to Poincaré (4.18) degenerates in this extreme limit but a modified map (with z^- shifted and rescaled) shows that the conjugacy class is elliptic/parabolic. The parabolic identification in the R -sector indeed breaks $SL(2, R)_R \rightarrow U(1)_R$. Regular particle states are elliptic on both sides [26,27], and so they similarly break both $SL(2, R)$'s.

4.3. The energy revisited

It is worth revisiting our notion of energy in AdS_2 in view of our discussion of global identifications in the reduction from 3D.

As we have argued, the dynamical theory is tied to the R sector. The relation (4.18) between z^- and the 2D time t implies the 2D temperature $T_{\text{phys}} = \frac{\epsilon}{2\pi\ell^2}$. This provides the 3D origin of the temperature (2.32) obtained directly in 2D.

In the usual extreme limit of BTZ, discussed in the end of the previous subsection, the coordinates are kept fixed as the temperature is lowered. Regular CFTs have a gap in the allowed conformal dimensions, so this extreme limit leaves just the ground state. The DLCQ limit scales the coordinates such that the periodicity increases. Then the energy of states are lowered along with the temperature is lowered and so a nontrivial sector remains. Alternatively, increasing the periodicity first we expect a continuous spectrum for the non-compact theory, with states remaining at arbitrarily low temperature.

In the DLCQ description the vacuum state encoded in the L sector is notoriously complicated. We can discern the temperature of this state by comparing the identifications (4.19) to the more conventional (4.17). We find the effective 2D temperature

$$T_+^{\text{FT}} = \frac{1}{2\pi} , \quad (4.20)$$

after compensating for the relative factor of 2 between dimensionless temperatures in 3D and 2D. This is the self-dual temperature, a strongly coupled value in the sense that the CFT is not well approximated by a free gas. The temperature (4.20) is the Frolov-Thorne temperature of the ground state, but this is not the sector we focus on in this paper.

The physical energy is encoded in the stress tensor dual to AdS_3 , projected along the direction that forms the boundary of AdS_2 . We have

$$T_{tt}^{3\text{D}} = \frac{4\epsilon^2}{\ell^2} \left[(r_+ - r_-)^{-2} T_{--}^{3\text{D}} + (r_+ + r_-)^{-2} T_{++}^{3\text{D}} \right] , \quad (4.21)$$

where we used the linear coordinate transformation (4.6) and the definition of ϵ (4.7). The standard 3D expressions computed from (3.3) are

$$T_{--}^{3\text{D}} = \frac{1}{32\pi G_3 L} (r_+ - r_-)^2 , \quad T_{++}^{3\text{D}} = \frac{1}{32\pi G_3 L} (r_+ + r_-)^2 , \quad (4.22)$$

and so

$$T_{tt}^{3\text{D}} = \frac{1}{4\pi G_3 L} \frac{\epsilon^2}{\ell^2} . \quad (4.23)$$

It is interesting that the two terms in (4.21) give the same contribution to the total energy (4.23). In other words, the total 3D energy receives equal contributions from the excitations (in the $R = -$ sector) and the inert condensate (in the $L = +$ sector).

5. Implications for Kerr/CFT

At this point we have computed boundary currents, central charge and the asymptotic symmetry group of AdS₂ gravity directly in 2D, and we have validated our results by comparison with a near DLCQ limit of 3D pure gravity. We can now apply the results to the thermodynamics of near extremal Kerr black holes.

First we review the thermodynamics of the Kerr solution. When manipulating thermodynamic formulae it is useful to introduce the Planck length l_P through

$$G_4 = l_P^2 . \quad (5.1)$$

In this notation, we are interested in near extremal Kerr black holes for which the mass

$$M = \frac{\sqrt{J}}{l_P} + E , \quad (5.2)$$

with excitation energy $E \ll \sqrt{J}/l_P$ (with $\ell^2 \sim 2Jl_P^2$ from (2.6)), or equivalently $E\ell/J \ll 1$. The excitation energy is related to the non-extremality parameter ϵ of the effective AdS₂ black hole as

$$\epsilon^2 \lambda^2 = 4\sqrt{J}E\ell_P^3 , \quad (5.3)$$

where $\lambda \ll 1$ is the scaling parameter defining the near extremal limit (details in Appendix A).

The black hole entropy for a general Kerr black hole is

$$S = \frac{A}{4G_4} = 2\pi(M^2 l_P^2 + \sqrt{M^4 l_P^4 - J^2}) . \quad (5.4)$$

For small excitation energy the Hawking temperature becomes

$$\begin{aligned} T_H &= \frac{1}{2\pi l_P} J^{-3/4} \sqrt{E l_P} \\ &= \frac{\epsilon \lambda}{2\pi \ell^2} , \end{aligned} \quad (5.5)$$

so we can write the excitation energy as

$$E = 4\pi^2 J^{3/2} T_H^2 l_P . \quad (5.6)$$

This is the same form as the high temperature behavior

$$E = \frac{\pi^2}{6} c T_H^2 R , \quad (5.7)$$

of a general CFT on a circle of radius R . It is this agreement in form that allows a description of the excitations in terms of a dual CFT.

We can understand some features of the dual CFT from our results in this paper. First, our result (2.35) for the central charge becomes

$$c = \frac{6\ell^2}{G_4} = 12J , \quad (5.8)$$

after inserting the AdS_2 radius (2.6) appropriate for the near horizon region of the extreme Kerr black hole. This is the same value that Kerr/CFT assigns to the ground state, but here it controls the excitations. We stress again that our result (2.35) for the central charge is intrinsic to 2D, although consistent with the Brown-Henneaux value (3.30) for AdS_3 through the map (3.21) to AdS_2 .

Comparison of (5.6) and (5.7) gives the scale R of the dual CFT as

$$R = 2\sqrt{J}l_P . \quad (5.9)$$

This is slightly bigger than the AdS_2 radius $R = \sqrt{2}\ell$ and consistent with the result reported in [28]⁶.

Now, the excitation energy E and the temperature T_H both vanish in the strict extremal limit $\lambda \rightarrow 0$ as measured by asymptotic observers using the time t' . However, we can rescale the physics in the manner made explicit for the 3D/2D reduction in the previous section. In the Kerr context we use $t = \lambda t'$ and the temperature $T = T_H/\lambda$ is finite. The conformal weight assigned to a state is finite *after* the rescaling and the extremal limit and it reads

$$h = \frac{ER}{\lambda^2} = \frac{c}{24} \left(\frac{2\pi T_H R}{\lambda} \right)^2 = \frac{c}{24} \frac{2\epsilon^2}{\ell^2} . \quad (5.10)$$

This is the value that exactly reproduces the entropy of the excitations in (5.4) in the near extremal limit, *i.e.*

$$S = 2\pi \left(\frac{c_L}{12} + \sqrt{\frac{c_R h}{6}} \right) + \dots . \quad (5.11)$$

with $c_{L,R} = c = 12J$. After rescaling the conformal weights we have $h \gg c$ in the black hole regime $\epsilon \gg \ell$ so that Cardy's formula applies.

⁶ The model in [28] allocates half the energy to the zero-modes, as suggested after (4.23). This doubles R to $\sqrt{8}\ell$, while the energy and the temperature of the excitations are 1/2 of their values given in this section.

In the AdS_2 theory, we employed the scale ℓ to measure energies in (2.33), but this is not the scale describing the breaking of conformal symmetry in (5.7). The appropriate comparison with the discussion in section 2.4 is energy per unit length

$$\frac{E}{R\lambda} = \frac{\pi^2}{6} cT^2 = \frac{T_{tt} + \mathcal{B}_t J_t}{\ell} . \quad (5.12)$$

The unambiguous point of comparison between the 4D geometry and AdS_2 is the time coordinate t . This ensures the agreement in (5.12), despite the ambiguity in the scale relevant for E in the 2D theory, due to the $SL(2, R)$ invariance of the theory.

6. Discussion

One of the highlights of this paper is the agreement between the entropy inferred from the study of the conformal symmetry acting on the quantum theory, and the entropy computed from the area law applied to the Kerr metric. As in many similar agreements of the sort, a key step in the computation is the presentation of the entropy from either point of view in the form of a Cardy's formula.

Now, in the context of the BTZ black hole it is an important point that the Cardy form of the entropy, as well as the numerical values of its various parameters, are in fact guaranteed by symmetries. The reported agreements are therefore *automatic*, in the sense that they follow from symmetries (see eg. [29]). It is interesting to ask whether this sort of automatic agreement extends to the AdS_2 gravity considered in this paper.

A good starting point is the free energy (on-shell action) of the $SL(2, R)$ invariant ground state (*ie* with $L_0 = 0$),

$$I_{\text{gs}} = \beta(L_0 - \frac{c}{24}) = \beta(-\frac{c}{24}) . \quad (6.1)$$

The black hole is the high temperature dual with free energy

$$I_{\text{bh}} = (2\pi)^2 T(-\frac{c}{24}) . \quad (6.2)$$

obtained from (6.1) by $2\pi\beta \rightarrow \frac{1}{2\pi\beta}$. This latter expression is equivalent to Cardy's formula, upon transforming to the micro-canonical ensemble.

The relation between ground state and highly excited modes in 2D CFT is of course very well known, but what we seek is the corresponding relation on the gravitational side, which then guarantees the accuracy of the accounting for the black hole entropy. In 3D

the geometrical origin of the duality is the large coordinate transformation exchanging the temporal and the azimuthal period in Euclidean gravity (see *eg.* [29]). In particular, the azimuthal circle is contractible in global AdS_3 while the temporal circle is contractible for the (non-rotating) BTZ black hole [30].

In our discussion of AdS_2 quantum gravity the description as an AdS_2 black hole applies for large $\epsilon^2/\ell^2 \gg 1$, which ensures $h_R \gg c$ such that the asymptotic form of the degeneracy applies. On the other hand, the $SL(2, R)_R$ invariant vacuum state is global AdS_2 , corresponding to $\epsilon^2/\ell^2 = -1$. The high-low temperature duality of the CFT relates these limits, and so provides the key ingredient for a gravitational justification of the Cardy formula, which in turn implies an automatic origin of the agreements.

The aspect of the agreement that remains mysterious is the geometrical interpretation of the duality symmetry. Ultimately it should follow from modular invariance in 3D but the details are nontrivial. For example the reduction is incompatible with the Euclidean continuation (see *eg.* (4.6)), the customary setting for modular invariance. It would be nice to understand this feature in complete detail.

Acknowledgments:

We thank M. Cheng, T. Hartman, A. Lepage-Jutier, A. Maloney and A. Strominger for discussions. CK and FL thank the Aspen Centre for Physics for hospitality during part of this work. The work of AC is supported in part by the National Science and Engineering Research Council of Canada. The work of CK is supported by the Fundamental Laws Initiative at the Center for the Fundamental Laws of Nature. The work of FL is supported in part by the US DoE.

Appendix A. Near horizon geometry of the Kerr black hole

The general Kerr solution is given by

$$ds^2 = - \frac{\Sigma \Delta}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta} dt'^2 + \Sigma \left[\frac{dr^2}{\Delta} + d\theta^2 \right] + \frac{\sin^2 \theta}{\Sigma} ((r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta) \left[d\phi' - \frac{2a\mu r}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta} dt' \right]^2, \quad (\text{A.1})$$

with

$$\Delta = (r - r_-)(r - r_+), \quad r_{\pm} = \mu \pm \sqrt{\mu^2 - a^2}, \quad (\text{A.2})$$

and

$$\Sigma = r^2 + a^2 \cos^2 \theta . \quad (\text{A.3})$$

In our notation $\mu = G_4 M$ and $a = J/M$ are length scales, while J is dimensionless. The near horizon region is isolated by introducing the coordinates

$$r = \frac{1}{2}(r_+ + r_-) + \lambda U , \quad t' = \frac{t}{\lambda} , \quad \phi' = \phi + \frac{t}{\lambda(r_+ + r_-)} . \quad (\text{A.4})$$

In the strict near horizon limit the dimensionless scaling parameter $\lambda \rightarrow 0$, with t, U, θ, ϕ fixed. In the extremal limit $r_+ = r_-$, the metric is known as NHEK geometry [31]. This limit is easily modified to maintain some energy above extremality. We need to take the limit while tuning the black hole parameters such that the scale ϵ defined through

$$\frac{1}{2}(r_+ - r_-) = \sqrt{\mu^2 - a^2} \equiv \epsilon \lambda , \quad (\text{A.5})$$

is kept fixed as $\lambda \rightarrow 0$. The resulting line element reads

$$ds^2 = \frac{1 + \cos^2 \theta}{2} \left[-\frac{U^2 - \epsilon^2}{\ell^2} dt^2 + \frac{\ell^2}{U^2 - \epsilon^2} dU^2 + \ell^2 d\theta^2 \right] + \ell^2 \frac{2 \sin^2 \theta}{1 + \cos^2 \theta} \left(d\phi + \frac{U}{\ell^2} dt \right)^2 , \quad (\text{A.6})$$

with

$$\ell^2 \equiv \frac{1}{2}(r_+ + r_-)^2 = 2\mu^2 , \quad (\text{A.7})$$

fixed. The (t, U) term in the square brackets is locally AdS_2 with radius of curvature ℓ but the global structure is modified into a black hole geometry with horizon located at $U = \epsilon$. The near extreme limit described here is the same considered in [32,33] (and a Kerr analogue of “Limit 2” in [22]).

Appendix B. Fefferman-Graham expansion in 2D

In this appendix we provide further details about the Fefferman-Graham expansion in 2D and some useful formulas. We write the metric and gauge fields as

$$\begin{aligned} h_{tt} &= h^{(0)} e^{2\rho/\ell} + h^{(2)} + h^{(4)} e^{-2\rho/\ell} + \dots , \\ \mathcal{B}_t &= \mathcal{B}^{(0)} e^{\rho/\ell} + \mathcal{B}^{(2)} + \mathcal{B}^{(4)} e^{-\rho/\ell} + \dots , \end{aligned} \quad (\text{B.1})$$

and take $\psi = 0$. We can relate the coefficients of the asymptotic expansion by using Maxwell’s eqn. This tells us that the field strength is proportional to the volume form

$$d\mathcal{B} = \ell \epsilon_{\text{AdS}_2} , \quad (\text{B.2})$$

and leads to

$$\begin{aligned}
h^{(0)} &= -\ell^2 \mathcal{B}^{(0)} \mathcal{B}^{(0)} , \\
h^{(2)} &= 2\ell^2 \mathcal{B}^{(0)} \mathcal{B}^{(4)} , \\
h^{(4)} &= -\ell^2 \left(\mathcal{B}^{(4)} \mathcal{B}^{(4)} - 6\mathcal{B}^{(0)} \mathcal{B}^{(8)} \right) , \\
h^{(6)} &= \ell^2 \left(10\mathcal{B}^{(0)} \mathcal{B}^{(12)} - 6\mathcal{B}^{(4)} \mathcal{B}^{(8)} \right) .
\end{aligned} \tag{B.3}$$

For the black hole solution (4.16), (2.5) the non-zero coefficients of the expansion are

$$h^{(0)} = -\frac{1}{4} , \quad h^{(2)} = \frac{\epsilon^2}{2\ell^2} , \quad h^{(4)} = -\frac{\epsilon^4}{4\ell^4} , \tag{B.4}$$

and

$$\mathcal{B}^{(0)} = \frac{1}{2\ell} , \quad \mathcal{B}^{(2)} = -\frac{\epsilon}{\ell^2} , \quad \mathcal{B}^{(4)} = \frac{\epsilon^2}{2\ell^3} . \tag{B.5}$$

The 3D boundary stress tensor is given by (3.3), with the extrinsic curvature defined as

$$\begin{aligned}
K_{ab} &= \frac{1}{2L} \partial_\eta \gamma_{ab} \\
&= \frac{1}{L} e^{2\eta} \gamma_{ab}^{(0)} - \frac{1}{L} e^{-2\eta} \gamma_{ab}^{(4)} + \dots ,
\end{aligned} \tag{B.6}$$

and its trace $K = \gamma^{ab} K_{ab}$, where

$$\gamma^{ab} = e^{-2\eta} \gamma^{(0)ab} - e^{-4\eta} \gamma_{cd}^{(2)} \gamma^{(0)ac} \gamma^{(0)bd} + \dots . \tag{B.7}$$

Using (B.6) and (B.7), the stress tensor (3.3) in terms of the boundary metric (3.5) is

$$T_{ab}^{3D} = \frac{1}{8\pi G_3 L} \left[\gamma_{ab}^{(2)} - \gamma_{ab}^{(0)} \text{Tr}(\gamma^{(2)}) + e^{-2\eta} \left(2\gamma_{ab}^{(4)} - \gamma_{ab}^{(2)} \text{Tr}(\gamma^{(2)}) - \gamma_{ab}^{(0)} \text{Tr}(\gamma^{(4)}) \right) + \dots \right] , \tag{B.8}$$

with $\text{Tr}(\gamma^{(n)}) = \gamma^{(0)ab} \gamma_{ab}^{(n)}$. In terms of the 2D fields (B.1) the boundary metric is given by (3.12), and the components of (B.8) reduce to

$$\begin{aligned}
T_{tt}^{3D} &= \frac{\ell^2}{8\pi G_3 L} \left[\mathcal{B}^{(2)} \mathcal{B}^{(2)} + 4\mathcal{B}^{(0)} \mathcal{B}^{(4)} + 12e^{-2\eta} \mathcal{B}^{(2)} \mathcal{B}^{(4)} + \mathcal{O}(e^{-4\eta}) \right] , \\
T_{t\theta}^{3D} &= \frac{\ell^2}{8\pi G_3 L} \left[\mathcal{B}^{(2)} + 6e^{-2\eta} \mathcal{B}^{(4)} + \mathcal{O}(e^{-4\eta}) \right] , \\
T_{\theta\theta}^{3D} &= \frac{\ell^2}{8\pi G_3 L} \left[1 + 4e^{-4\eta} \frac{\mathcal{B}^{(4)}}{\mathcal{B}^{(0)}} + \mathcal{O}(e^{-6\eta}) \right] ,
\end{aligned} \tag{B.9}$$

where we used (B.3) to write $h^{(n)}$ as a function of $\mathcal{B}^{(n)}$.

References

- [1] A. Strominger, “AdS(2) quantum gravity and string theory,” JHEP **9901**, 007 (1999) [arXiv:hep-th/9809027].
- [2] A. Sen, “Quantum Entropy Function from AdS(2)/CFT(1) Correspondence,” arXiv:0809.3304 [hep-th].
- [3] V. Balasubramanian and P. Kraus, “A stress tensor for anti-de Sitter gravity,” Commun. Math. Phys. **208**, 413 (1999) [arXiv:hep-th/9902121].
- [4] R. Emparan, C. V. Johnson and R. C. Myers, “Surface terms as counterterms in the AdS/CFT correspondence,” Phys. Rev. D **60**, 104001 (1999) [arXiv:hep-th/9903238].
- [5] S. de Haro, S. N. Solodukhin and K. Skenderis, “Holographic reconstruction of space-time and renormalization in the AdS/CFT correspondence,” Commun. Math. Phys. **217**, 595 (2001) [arXiv:hep-th/0002230].
- [6] K. Skenderis, “Lecture notes on holographic renormalization,” Class. Quant. Grav. **19**, 5849 (2002) [arXiv:hep-th/0209067].
- [7] A. Castro, D. Grumiller, F. Larsen and R. McNees, “Holographic Description of AdS₂ Black Holes,” JHEP **0811**, 052 (2008) [arXiv:0809.4264 [hep-th]].
- [8] A. Castro and F. Larsen, “Near Extremal Kerr Entropy from AdS₂ Quantum Gravity,” JHEP **0912**, 037 (2009) [arXiv:0908.1121 [hep-th]].
- [9] R. K. Gupta and A. Sen, “Ads(3)/CFT(2) to Ads(2)/CFT(1),” JHEP **0904**, 034 (2009) [arXiv:0806.0053 [hep-th]].
- [10] V. Balasubramanian, J. de Boer, M. M. Sheikh-Jabbari and J. Simon, “What is a chiral 2d CFT? And what does it have to do with extremal black holes?,” arXiv:0906.3272 [hep-th].
- [11] M. Cadoni and S. Mignemi, “Asymptotic symmetries of AdS(2) and conformal group in $d = 1$,” Nucl. Phys. B **557**, 165 (1999) [arXiv:hep-th/9902040].
- [12] J. Navarro-Salas and P. Navarro, “AdS(2)/CFT(1) correspondence and near-extremal black hole entropy,” Nucl. Phys. B **579**, 250 (2000) [arXiv:hep-th/9910076].
- [13] J. H. Cho, T. Lee and G. W. Semenoff, “Two dimensional anti-de Sitter space and discrete light cone quantization,” Phys. Lett. B **468**, 52 (1999) [arXiv:hep-th/9906078].
- [14] M. Brigante, S. Cacciatori, D. Klemm and D. Zanon, “The Asymptotic Dynamics of two-dimensional (anti-)de Sitter Gravity,” JHEP **0203**, 005 (2002) [arXiv:hep-th/0202073].
- [15] M. Astorino, S. Cacciatori, D. Klemm and D. Zanon, “AdS(2) supergravity and superconformal quantum mechanics,” Annals Phys. **304**, 128 (2003) [arXiv:hep-th/0212096].
- [16] H. L. Verlinde, “Superstrings on AdS(2) and superconformal matrix quantum mechanics,” arXiv:hep-th/0403024.

- [17] T. Azeyanagi, T. Nishioka and T. Takayanagi, “Near Extremal Black Hole Entropy as Entanglement Entropy via AdS₂/CFT₁,” *Phys. Rev. D* **77**, 064005 (2008) [arXiv:0710.2956 [hep-th]].
- [18] M. Guica, T. Hartman, W. Song and A. Strominger, “The Kerr/CFT Correspondence,” arXiv:0809.4266 [hep-th].
- [19] Y. Matsuo, T. Tsukioka and C. M. Yoo, “Another Realization of Kerr/CFT Correspondence,” arXiv:0907.0303 [hep-th].
- [20] Y. Matsuo, T. Tsukioka and C. M. Yoo, “Yet Another Realization of Kerr/CFT Correspondence,” arXiv:0907.4272 [hep-th].
- [21] J. Rasmussen, “Isometry-preserving boundary conditions in the Kerr/CFT correspondence,” arXiv:0908.0184 [hep-th].
- [22] J. M. Maldacena, J. Michelson and A. Strominger, “Anti-de Sitter fragmentation,” *JHEP* **9902**, 011 (1999) [arXiv:hep-th/9812073].
- [23] M. Spradlin and A. Strominger, “Vacuum states for AdS(2) black holes,” *JHEP* **9911**, 021 (1999) [arXiv:hep-th/9904143].
- [24] A. Sen, “Arithmetic of Quantum Entropy Function,” arXiv:0903.1477 [hep-th].
- [25] T. Hartman and A. Strominger, “Central Charge for AdS₂ Quantum Gravity,” *JHEP* **0904**, 026 (2009) [arXiv:0803.3621 [hep-th]].
- [26] J. M. Maldacena and A. Strominger, “AdS(3) black holes and a stringy exclusion principle,” *JHEP* **9812**, 005 (1998) [arXiv:hep-th/9804085].
- [27] E. J. Martinec, “Conformal field theory, geometry, and entropy,” arXiv:hep-th/9809021.
- [28] M. Cvetič and F. Larsen, “Greybody Factors and Charges in Kerr/CFT,” *JHEP* **0909**, 088 (2009) [arXiv:0908.1136 [hep-th]].
- [29] P. Kraus and F. Larsen, “Microscopic Black Hole Entropy in Theories with Higher Derivatives,” *JHEP* **0509**, 034 (2005) [arXiv:hep-th/0506176].
- [30] R. Dijkgraaf, J. M. Maldacena, G. W. Moore and E. P. Verlinde, “A black hole farey tail,” arXiv:hep-th/0005003.
- [31] J. M. Bardeen and G. T. Horowitz, “The extreme Kerr throat geometry: A vacuum analog of AdS(2) x S(2),” *Phys. Rev. D* **60**, 104030 (1999) [arXiv:hep-th/9905099].
- [32] A. J. Amsel, G. T. Horowitz, D. Marolf and M. M. Roberts, “No Dynamics in the Extremal Kerr Throat,” arXiv:0906.2376 [hep-th].
- [33] O. J. C. Dias, H. S. Reall and J. E. Santos, “Kerr-CFT and gravitational perturbations,” arXiv:0906.2380 [hep-th].